Student Name ______________________________

Teacher's Name: ______________________________

Sydney Technical High School

Extension 2 Mathematics

TRIAL HSC

August 2020

General Instructions

- Reading time – 10 minutes
- Working time – 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I

10 Marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1. If $z = 2e^{-\frac{\pi}{3}i}$, then which of the following is purely real?

(A) $\bar{z}$

(B) z^3

(C) z^i

(D) $3z$

2. A particle's acceleration $\ddot{x}$ ms^{-2} is defined by $\ddot{x} = -3x$, where x is the displacement in metres. How long, in seconds, does it take to travel between the endpoints of the motion?

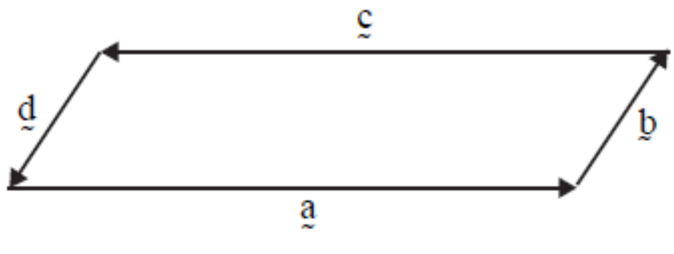
(A) $\dfrac{\pi\sqrt{3}}{3}$

(B) $9\pi\sqrt{3}$

(C) $3\pi\sqrt{3}$

(D) $\pi\sqrt{3}$

3. In the parallelogram, $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$



Which one of the following statements is true?

(A) $\underset{\sim}{a} = 2\underset{\sim}{b}$

(B) $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

(C) $\underset{\sim}{a} + \underset{\sim}{c} = 0$

(D) $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

4. If $z = 3 - 4i$, then $\frac{1}{1-z}$ is equal to:

(A) $\dfrac{-1-2i}{10}$

(B) $\dfrac{-1+2i}{10}$

(C) $\dfrac{-1-i}{6}$

(D) $\dfrac{-1+i}{6}$

5. Assume that a and b are negative real numbers with $a > b$. Which of the following might be false?

(A) $\dfrac{1}{a-b} < 0$

(B) $\dfrac{a}{b} - \dfrac{b}{a} < 0$

(C) $a + b > 2b$

(D) $2a > 3b$

6. If the vectors $\underset{\sim}{a} = m\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m\underset{\sim}{i} + m\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular, then:

(A) $m = -6$ or $m = 2$

(B) $m = -2$ or $m = 6$

(C) $m = -2$ or $m = 0$

(D) $m = -1$ or $m = 1$

7. Suppose that both x and y are odd. Which of the following statements is true?

(A) $x + y$ is odd

(B) $x - y$ is even

(C) $3x + 5y$ is odd

(D) xy is even

8. Using a suitable substitution, $\int_{1}^{e^3} \frac{(\ln(x))^3}{x}\, dx$ may be expressed completely in terms of u as:

(A) $\int_{0}^{3} \frac{u^3}{e^u}\, \mathrm{du}$

(B) $\int_{0}^{e^3} u^3 du$

(C) $\int_{0}^{3} u^3 du$

(D) $\int_{1}^{\ln(3)} u^3 du$

9. Let the complex number z satisfy the equation $|z + 4i| = 3$. What are the greatest and least values of $|z + 3|$?

(A) 8 and 2

(B) 5 and 2

(C) 8 and 3

(D) 8 and 5

10. P, Q and R are three collinear points with position vectors $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively. Q lies between P and R. If $2|\overrightarrow{QR}| = |\overrightarrow{PQ}|$, then $\underset{\sim}{r}$ is equal to:

(A) $\frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

(B) $\frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

(C) $\frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

(D) $\frac{1}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{q}$

Section II

Total marks – 90
Attempt Question 11-16
Allow about 2 hours and 45 minutes for this section

Begin each question on a NEW page

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Begin a NEW page.

a) What is the unit vector that has the same direction as $\underset{\sim}{v} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$ **2**

b) If $z = 6e^{i\frac{\pi}{3}}$

i. Rewrite z in modulus argument form **1**

ii. Simplify $(z)^4$ **2**

iii. Find $\arg(iz)$ **2**

c) Integrate:

i. $\displaystyle\int \frac{x+1}{x^2+2x+5}\,dx$ **2**

ii. $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$ **2**

d) Popularised on the internet, different types of dogs are given “names”. Three of these names are given as the following propositions.

p: it is not a doggo
q: it is a floofer
r: it is a woofer

i. Write down in words, $p \Rightarrow \text{not } r$ **2**

ii. Write down the converse of $p \Rightarrow (\text{not } q \text{ and } r)$ **1**

iii. Write down the contrapositive of $p \Rightarrow \text{not } q$ **1**

End of Question 11

Question 12 (15 marks) Begin a NEW page.

a) Consider the complex numbers $z_1 = 1 + i$ and $z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

i. Write z_1 in the form $r(cos\theta + isin\theta)$ **2**

ii. Find the modulus of $z_1 z_2$ **1**

iii. By considering the expansion of $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$, or otherwise, write $z_1 z_2$ in the form $z = a + bi$ where $a, b \in \mathbb{R}$ **3**

iv. Hence find the value of $\tan\frac{5\pi}{12}$ in the form $c + d\sqrt{3}$, where $c, d \in \mathbb{Z}$ **2**

v. Find the smallest value $p > 0$ such that $(z_2)^p$ is a positive real number **1**

b) Find $\int x^2 e^x \, dx$ **2**

c) A mass has acceleration a ms^{-2} given by $a = v^2 - 3$, where v ms^{-1} is the velocity of the mass when it has a displacement of x metres from the origin. **4**
Find v in terms of x given that $v = -2$ where $x = 1$.

End of Question 12

Question 13 (15 marks) Begin a NEW page.

a) The equations of intersecting lines L and M are given below with respect to a fixed origin O. **3**

$$L: \underset{\sim}{r} = 11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k} + \lambda\left(-2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}\right)$$
$$M: \underset{\sim}{r} = -5\underset{\sim}{i} + 11\underset{\sim}{j} + 1\underset{\sim}{k} + \mu\left(p\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

where λ and μ are parameters and p is a constant.
If L and M are perpendicular, what is the value of p?

b) Prove by induction that **3**

$$\frac{1}{n!} < \frac{1}{2^{n-1}}$$

for $n \geq 3, n \in \mathbb{Z}^+$

c) Let ω be a non-real cube root of unity

i. Show that $1 + \omega + \omega^2 = 0$ **1**

ii. Hence evaluate $(1 - 3\omega + \omega^2)(1 + \omega - 8\omega^2)$ **3**

d) i. Show that **2**

$$\frac{\tan x}{7\sin^2 x + 5\cos^2 x} = \frac{\tan x \sec^2 x}{7\tan^2 x + 5}$$

ii. Hence, by setting $s = \tan x$, or otherwise, find: **3**

$$\int_0^{\frac{\pi}{3}} \frac{\tan x \, dx}{7\sin^2 x + 5\cos^2 x}$$

End of Question 13

Question 14 (15 marks) Begin a NEW page.

a) By writing $\dfrac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)}$ in the form $\dfrac{Ax + 1}{x^2 + 1} + \dfrac{B}{x - 2}$ find **3**

$$\int \frac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)}\, dx$$

b) If a, b and $x \geq 0$,

i. Show that $a^2 + b^2 \geq 2ab$ **1**

ii. Hence, show that $\dfrac{x}{x^2 + 4} \leq \dfrac{1}{4}$ **2**

iii. By integrating both sides between the limits of $x = 0$ and $x = X$, show that **2**

$$e^{\frac{X}{2}} \geq \frac{X^2}{4} + 1 \text{ where } X \geq 0$$

c) A particle is undergoing simple harmonic motion about $x = 0$. At time t seconds the displacement x in cm is given by

$$x = \sqrt{3}\sin 3t - \cos 3t$$

i. Write $x = \sqrt{3}\sin 3t - \cos 3t$ in the form of $x = A\sin(nt - \theta)$, where $A > 0$ **2**

ii. Find the period of the motion **1**

iii. When does the particle first reach maximum speed after time $t = 0$? **2**

iv. How long will it take for the particle to return to its original position, and find its acceleration at that point. **2**

End of Question 14

Question 15 (15 marks) Begin a NEW page.

a) If *xyz* represents a three digit number (not the product of *x*, *y* and *z*), show that if $x + z = y$ then the number is divisible by 11. (*x*, *y* and *z* are positive integers) **2**

b) A particle is moving under simple harmonic motion where $v = 9\sqrt{3 - x^2}$. Find the centre of motion. **2**

c) The points A and B have position vectors given by:

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

i. Find an expression for the vector $\overrightarrow{AB}$ in the form of $x_1 \underset{\sim}{i} + y_1 \underset{\sim}{j} + z_1 \underset{\sim}{k}$ **1**

ii. Show that the cosine of the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ is $\frac{4}{9}$ **2**

iii. Hence find the exact value of the area of ΔOAB **3**

d) i. Show that **1**

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

ii. Consider $f(x) = \sin ax$, where a is a constant. **4**

Prove by mathematical induction that

$$f^{(n)}(x) = a^n \sin(ax + \frac{n\pi}{2})$$

where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

End of Question 15

Question 16 (15 marks) Begin a NEW page.

a) Let $I_n = \int_0^1 x^n e^{-x} dx$

i. Show that $I_n = nI_{n-1} - \frac{1}{e}$ **2**

ii. Hence or otherwise, find the exact value of $I_3 = \int_0^1 x^3 e^{-x} dx$ **2**

b) The point P representing the complex number z moves on the Argand diagram so that $|z| = |z - 6 + 4i|$

i. Find the Cartesian equation that describes the locus of P **2**

ii. Graph the locus neatly on a number plane. **1**

iii. Hence find the minimum value of $|z|$ **1**

Question 16 continues on the next page

c) The position vectors of the points A, B and C are $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively, relative to an origin O. The following diagram shows the triangle ABC and the points M, R, S and T.

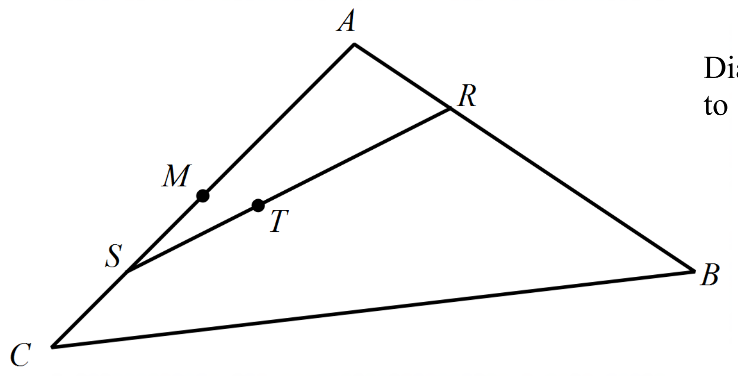


Diagram NOT to scale

M is the midpoint of $\overrightarrow{AC}$
R is on $\overrightarrow{AB}$ such that $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{AB}$
S is on $\overrightarrow{AC}$ such that $\overrightarrow{AS} = \frac{2}{3}\overrightarrow{AC}$
T is a point on $\overrightarrow{RS}$ such that $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$

i. Express $\overrightarrow{AM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ **1**

ii. Hence show $\overrightarrow{BM} = \frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}$ **1**

iii. Show that $\overrightarrow{RT} = -\frac{2}{9}\underset{\sim}{a} - \frac{2}{9}\underset{\sim}{b} + \frac{4}{9}\underset{\sim}{c}$ **2**

iv. Prove that T lies on $\overrightarrow{BM}$ **3**

End of Exam

Extension 2 Mathematics Trial Solutions

Section 1.

1. B	6. A
2. A	7. B
3. C	8. C
4. A	9. A
5. A	10. A

1. $(2e^{i\pi/3})^3$

$= 8e^{i\pi}$

Purely real.

2. $n = \sqrt{3}$

$T = 2\pi/\sqrt{3}$

Max → min is $\frac{1}{2}T$

$\therefore \pi/\sqrt{3} \times \sqrt{3}/\sqrt{3}$

$= \sqrt{3}\pi/3$ seconds

3. $\underset{\sim}{a} = -\underset{\sim}{c}$

$\underset{\sim}{a} + \underset{\sim}{c} = 0$

4. $\dfrac{1}{1-(3-4i)} = \dfrac{1}{-2+4i}$

$\dfrac{1}{-2+4i} \times \dfrac{-2-4i}{-2-4i}$

$= \dfrac{-2(1+2i)}{4+16}$

$= \dfrac{1+2i}{10}$

5. $a > b$

$a - b > 0$

$\dfrac{1}{a-b} > 0$

6. $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$m^2 + 4m - 12 = 0$

$(m-2)(m+6) = 0$

7. $x = 2m+1$, $y = 2n+1$

$x - y = 2m+1-(2n+1)$

$= 2m - 2n$

$= 2(m-n)$

8. let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$x = e^3 \Rightarrow u = 3$

$x = 1 \Rightarrow u = 0$

$\therefore I = \int_0^3 u^3\, du$

9. $|z - (-4i)| = 3$

is a circle.

where $|z+3| = |z-(-3)|$

magnitude of a vector

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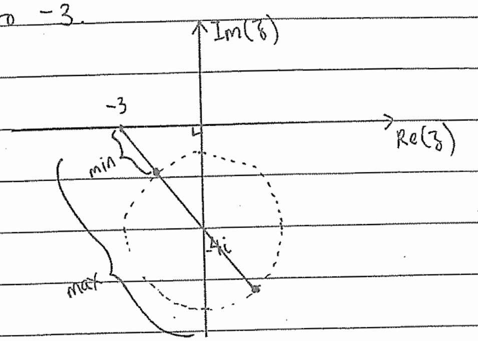
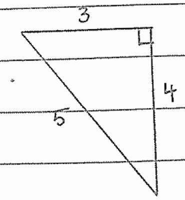
joining point on the circle to -3.



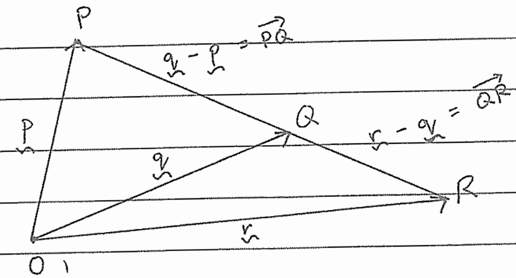


min value : $5-3$

max value : $5+3$

↖ radius

10.



$2(\underset{\sim}{r} - \underset{\sim}{q}) = \underset{\sim}{q} - \underset{\sim}{p}$

$2\underset{\sim}{r} = 3\underset{\sim}{q} - \underset{\sim}{p}$

$\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

Section II.

11. a) $|\underset{\sim}{v}| = \sqrt{6^2+3^2+1^2}$

$\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{46}}\begin{pmatrix}6\\3\\-1\end{pmatrix}$

b) i. $z = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ii. $z^4 = \left[6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^4$

By De Moivre's Theorem.

$z^4 = 6^4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 6^4\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

OR $6^4\left(\text{cis}\left(-\frac{2\pi}{3}\right)\right)$

iii. $\arg(iz) = \arg(i) + \arg(z)$

$= \frac{\pi}{2} + \frac{\pi}{3}$

$= \frac{5\pi}{6}$

c) i. $\int \frac{x+1}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx$

$= \frac{1}{2}\ln\left|x^2+2x+5\right| + C$

ii. $\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x^2+2x+1)+4}\,dx$

$= \int \frac{1}{(x+1)^2+4}\,dx$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

d) i. If it is not a doggo, then it implies that it is not a woofer.

ii. (not q and r) $\Rightarrow p$

iii not (not q) $\Rightarrow$ not p

$q \Rightarrow$ not p

12. a) i. $z_1 = 1 + i$

$|z_1| = \sqrt{2}$

$\arg(z_1) = \frac{\pi}{4}$

$\therefore z_1 = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

ii. $|z_1 z_2| = |z_1||z_2|$

$= 2\times\sqrt{2}$

$= 2\sqrt{2}$

iii. $z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)\right)$

$\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4}$

$= \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$

$\therefore z_1 z_2 = 2\sqrt{2}\left(\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} + i\left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right)\right)$

$= \sqrt{3} - 1 + i(1+\sqrt{3})$

iv. $\tan\frac{5\pi}{12} = \frac{\sin\frac{5\pi}{12}}{\cos\frac{5\pi}{12}}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \frac{3+2\sqrt{3}+1}{3-1}$

$= 2+\sqrt{3}$

v. $(z_2)^p$ positive real. $\cos\frac{p\pi}{6} > 0$

$\sin\frac{p\pi}{6} = 0$

$\therefore p = 12$

b) $\int x^2 e^x \, dx$

$= [x^2 e^x] - \int 2x e^x \, dx$

$= x^2 e^x - \left([2x e^x] - \int 2e^x \, dx \right)$

$= x^2 e^x - 2x e^x + 2e^x + C$

c) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$v^2 - 3 = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$2v^2 - 6 = \frac{dv^2}{dx}$

$\frac{1}{2v^2 - 6} = \frac{dx}{dv^2}$

$\int \frac{dv^2}{2v^2 - 6} = \int dx$

$x = \frac{1}{2}\ln|2v^2 - 6| + C$

when $x = 1$, $v = -2$

$1 = \frac{1}{2}\ln|8 - 6| + C$

$1 = \frac{1}{2}\ln 2 + C$

$C = 1 - \frac{1}{2}\ln 2$

$x = \frac{1}{2}\ln(2v^2 - 6) + 1 - \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln|v^2 - 3| + 1$

$2(x-1) = \ln|v^2 - 3|$

$e^{2(x-1)} = v^2 - 3$

$v^2 = e^{2(x-1)} + 3$

$v = \pm\sqrt{e^{2(x-1)} + 3}$

but as $x = 1$, $v = -2$

$\Rightarrow v = -\sqrt{e^{2(x-1)} + 3}$ only.

OR

$v^2 - 3 = v\frac{dv}{dx}$

$\frac{v^2 - 3}{v} = \frac{dv}{dx}$

$\int \frac{v}{v^2 - 3} \, dv = \int dx$

$\frac{1}{2}\ln|v^2 - 3| = x + C$

when $x = 1$, $v = -2$

$C = -1$

$\Rightarrow \frac{1}{2}\ln|v^2 - 3| = x - 1$

13. a) If perpendicular, direction vectors must be perpendicular

i.e. dot product zero.

$(-2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}) \cdot (p\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}) = 0$

$-2p + 2 - 8 = 0$

$-2p = 6$

$p = -3$

b) $\frac{1}{n!} < \frac{1}{2^{n-1}}$, $n \geq 3$, $n \in \mathbb{Z}^+$

Show true for $n = 3$.

LHS $= \frac{1}{3!}$

$= \frac{1}{6}$

RHS $= \frac{1}{2^{3-1}}$

$= \frac{1}{4} > \frac{1}{6}$

$\therefore$ True for $n = 3$.

Assume true for some $k \in n$

i.e. $\frac{1}{k!} < \frac{1}{2^{k-1}}$

Now prove true for $n = k+1$

i.e. Prove $\frac{1}{(k+1)!} < \frac{1}{2^{k+1-1}}$

From the assumption:

$\frac{1}{k!} \times \frac{1}{k+1} < \frac{1}{2^{k-1}} \times \frac{1}{k+1}$

$\frac{1}{(k+1)!} < \frac{1}{2^{k-1}(k+1)}$

$< \frac{2}{2^{k-1}(k+1) \times 2}$

$< \frac{2}{2^{k+1-1}(k+1)}$

$< \frac{1}{2^{k+1-1}} \times \frac{2}{k+1}$

$\frac{1}{(k+1)!} < \frac{1}{2^{k+1-1}}$ $\quad 0 < \frac{2}{k+1} < 1$ for $k \geq 3$

$\therefore$ True for $n = k+1$, given true for $n = k$

Since result is true for $n = 3$, it follows that it will

OR:

LHS $= \frac{1}{(k+1)!}$

$= \frac{1}{k!(k+1)}$

$< \frac{1}{2^{k-1}} \cdot \frac{1}{(k+1)}$ assump.

$= \frac{1}{k2^{k-1} + 2^{k+1}}$, $k \geq 3$

$< \frac{1}{2.2^{k-1} + 2^{k-1}}$

$= \frac{1}{3.2^{k-1}}$

$< \frac{1}{2.2^{k-1}}$

$= \frac{1}{2^{k-1+1}}$

$= \frac{1}{2^{k+1-1}}$

$\therefore \frac{1}{(k+1)!} < \frac{1}{2^{k+1-1}}$

also be true for $n = 3+1 = 4$, $n = 4+1 = 5$ and so on for all $n \geq 3$ integers.

c) i. $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

Now ω is a non-real root of unity.

i.e. $(\omega - 1)(\omega^2 + \omega + 1) = 0$ but $\omega \neq 1$

$\therefore \omega^2 + \omega + 1 = 0$.

ii. $(1 - 3\omega + \omega^2)(1 + \omega - 8\omega^2)$

$= ((1+\omega^2) - 3\omega)((1+\omega) - 8\omega^2)$

$= (-\omega - 3\omega)(-\omega^2 - 8\omega^2)$

$= -4\omega \times -9\omega^2$

$= 36\omega^3$ but $\omega^3 = 1$

$= 36$

d) i. $LHS = \dfrac{\tan x}{7\sin^2 x + 5\cos^2 x}$

$= \dfrac{\frac{\tan x}{\cos^2 x}}{7\frac{\sin^2 x}{\cos^2 x} + 5\frac{\cos^2 x}{\cos^2 x}}$

$= \dfrac{\tan x \sec^2 x}{7\tan^2 x + 5}$

$= RHS$.

ii. $\displaystyle\int_0^{\pi/3} \frac{\tan x}{7\sin^2 x + 5\cos^2 x}\,dx$

$= \displaystyle\int_0^{\pi/3} \frac{\tan x \sec^2 x}{7\tan^2 x + 5}\,dx$

from part i)

Let $s = \tan x$

$ds = \sec^2 x\,dx$

when $x = \pi/3$, $s = \sqrt{3}$

$x = 0$, $s = 0$

$\therefore I = \displaystyle\int_0^{\sqrt{3}} \frac{s\,ds}{7s^2 + 5}$

$= \dfrac{1}{14}\Big[\ln|7s^2 + 5|\Big]_0^{\sqrt{3}}$

$= \dfrac{1}{14}(\ln 26 - \ln 5)$

$= \dfrac{1}{14}\ln \dfrac{26}{5}$

14. a) $\frac{5x^2-3x+1}{(x^2+1)(x-2)} = \frac{Ax+1}{x^2+1} + \frac{B}{x-2}$

$5x^2-3x+1 = (Ax+1)(x-2) + B(x^2+1)$

let $x=2$

$15 = B(5)$

$B = 3$

let $x = 1$

$3 = (A+1)(-1) + 3(2)$

$-3 = -A-1$

$-2 = -A$

$A = 2$

$I = \int \frac{2x+1}{x^2+1} + \frac{3}{x-2}\,dx$

$= \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} + \frac{3}{x-2}\,dx$

$= \ln|x^2+1| + \tan^{-1}x + 3\ln|x-2| + C$

$= \ln|(x^2+1)(x-2)^3| + \tan^{-1}x + C$

b) i $(a-b)^2 \geq 0$

$a^2-2ab+b^2 \geq 0$

$a^2+b^2 \geq 2ab$

ii. let $a = x$, $b = 2$

$x^2+4 \geq 4x$

$\frac{1}{x^2+4} \leq \frac{1}{4x}$, x is positive so

$\frac{x}{x^2+4} \leq \frac{1}{4}$

iii. $\int_0^x \frac{x}{x^2+4}\,dx \leq \int_0^x \frac{1}{4}\,dx$

$\left[\frac{1}{2}\ln|x^2+4|\right]_0^x \leq \left[\frac{1}{4}x\right]_0^x$

$\frac{1}{2}\ln|x^2+4| - \frac{1}{2}\ln 4 \leq \frac{1}{4}x$

$\frac{1}{2}\ln\frac{x^2+4}{4} \leq \frac{x}{4}$

$\ln\frac{x^2+4}{4} \leq \frac{x}{2}$

$e^{\ln\left(\frac{x^2+4}{4}\right)} \leq e^{x/2}$

$\frac{x^2}{4}+1 \leq e^{x/2}$

c) i $x = \sqrt{3}\sin 3t - \cos 3t$

$x = A\sin 3t\cos\theta - A\cos 3t\sin\theta$

$\Rightarrow A\cos\theta = \sqrt{3}$

$A\sin\theta = 1$

$A = 2$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \pi/6$

$\therefore x = 2\sin(3t - \pi/6)$

ii. $T = 2\pi/n$, $n = 3$

$T = 2\pi/3$

iii. Max speed at $x=0$.

$2\sin(3t-\pi/6) = 0$

i.e. $3t - \pi/6 = 0, \pi, 2\pi, \ldots$

Take $3t - \pi/6 = 0$

$3t = \pi/6$

$t = \pi/18$ seconds.

iv. Original position : $t=0$.

$x = 2\sin(-\pi/6)$

$= 2 \times -\frac{1}{2}$

$= -1$

$\therefore -1 = 2\sin(3t - \pi/6)$

$-\frac{1}{2} = \sin(3t - \pi/6)$

$\Rightarrow 3t - \pi/6 = -\pi/6$ or $\frac{7\pi}{6}$ or ....

↑ this is $t=0$.

$\therefore 3t - \pi/6 = 7\pi/6$

$3t = 8\pi/6$

$t = \frac{4\pi}{3 \times 3}$

$= 4\pi/9$ seconds

Acceleration at the point $t = 4\pi/9$, $x=-1$

$\ddot{x} = -n^2 x$

$= -3^2 \times -1$

$= 9$ cm/s^2.

15. a) $xyz = 100x + 10y + z$

but $y = x + z$

$\therefore xyz = 100x + 10(x+z) + z$

$= 110x + 11z$

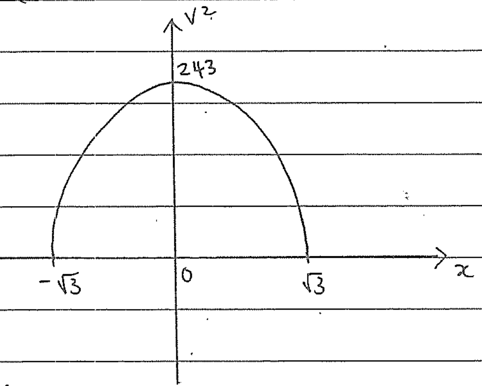
$= 11(10x + z)$

$\therefore$ divisible by 11 as x, z are positive integers.

b) $v = 9\sqrt{3 - x^2}$

centre is where v is max, v^2 is max.

$v^2 = 81(3 - x^2)$



Centre of motion at $x = 0$.

c) i. $\vec{AB} = -1\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k} - (1\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$

$= -2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

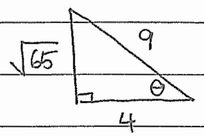
ii. $\cos\theta = \dfrac{\vec{OA} \cdot \vec{AB}}{|\vec{OA}| \times |\vec{AB}|}$

$= \dfrac{-2 + 2 + 4}{\sqrt{9} \times \sqrt{9}}$

$= \dfrac{4}{9}$

iii. $\triangle OAB = \frac{1}{2}|\vec{OA}| \times |\vec{AB}| \sin\theta$

$= \frac{1}{2} \times 9 \times \sin\theta$



$\triangle OAB = \frac{1}{2} \times 9 \times \frac{\sqrt{65}}{9}$

$= \frac{1}{2}\sqrt{65}\ u^2$

d) i. $\sin(\theta + \frac{\pi}{2}) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$

$= \sin\theta \times 0 + \cos\theta \times 1$

$= \cos\theta$

ii. $f^{(n)}(x) = a^n \sin(ax + \frac{n\pi}{2})$

Show true for $n=1$

$f'(x) = a\cos ax$

By formula: $f'(x) = a^1 \sin(ax + \frac{\pi}{2})$

$= a\cos ax$ (by part i.)

Assume true for $n=k$

i.e. $f^k(x) = a^k \sin(ax + \frac{k\pi}{2})$

Prove true for $n=k+1$

i.e. $f^{(k+1)}(x) = a^{(k+1)} \sin(ax + \frac{k\pi}{2})$

From the assumption:

$f^{(k)}(x) = a^k \sin(ax + \frac{k\pi}{2})$

differentiate both sides

$f^{(k+1)}(x) = a^k \cos(ax + \frac{k\pi}{2}) \times a$

$= a^{k+1} \cos(ax + \frac{k\pi}{2})$

$= a^{k+1} \sin(ax + \frac{k\pi}{2} + \frac{\pi}{2})$ (by part i.)

$= a^{k+1} \sin(ax + (k+1)\frac{\pi}{2})$

$\therefore$ true for $n=k+1$ provided $n=k$ is true.

Since true for $n=1$, then it follows true for $n=1+1=2$, $n=2+1=3$ and so on for all integer $n \geq 1$.

16. a) $I_n = \int_0^1 x^n e^{-x}\,dx$

i. $I_n = \left[x^n \times -e^{-x}\right]_0^1 - \int_0^1 nx^{n-1}(-e^{-x})\,dx$

$= 1^n \times (-e^{-1}) - 0^n \times (-e^{0}) + \int_0^1 nx^{n-1}e^{-x}\,dx$

$= -\frac{1}{e} + n\int_0^1 x^{n-1}e^{-x}\,dx$

$= -\frac{1}{e} + nI_{n-1}$

ii. $I_3 = 3I_2 - \frac{1}{e}$

$= 3\left(2I_1 - \frac{1}{e}\right) - \frac{1}{e}$

$= 6I_1 - 4/e$

$= 6\left(I_0 - 1/e\right) - 4/e$

$= 6I_0 - 10/e$

$= 6\int_0^1 x^0 e^{-x}\,dx - 10/e$

$= 6\int_0^1 e^{-x}\,dx - \frac{10}{e}$

$= 6\left[-e^{-x}\right]_0^1 - \frac{10}{e}$

$= 6\left(-e^{-1} + 1\right) - \frac{10}{e}$

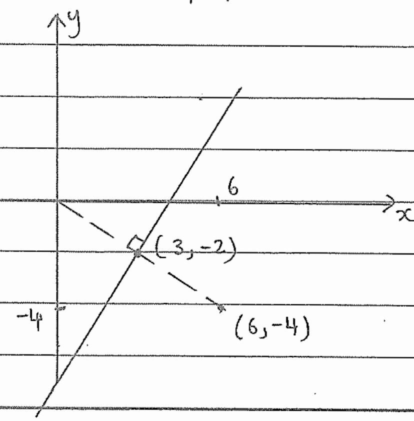
$= -6/e + 6 - 10/e$

$= 6 - 16/e$

b) i. $|z| = |z - 6 + 4i|$

$= |z - (6 - 4i)|$ perpendicular bisector

ii.



i. $M = (3, -2)$

$m = -4/6 \Rightarrow$ perpendicular gives $3/2$

$\therefore\ y + 2 = 3/2\,(x - 3)$

$y = 3/2\,x - 9/2 - 2$

$= 3/2\,x - 13/2$

OR $3x - 2y - 13 = 0$

iii. Minimum value by Pythagoras' Theorem:

$\sqrt{3^2 + 2^2} = \sqrt{13}$

c) i. $\vec{AM} = \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a})$

ii. $\vec{BM} = \frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}$

$= \vec{OM} - \vec{OB}$

$= \vec{OA} + \vec{AM} - \vec{OB}$

$= \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a}) - \underset{\sim}{b}$

$= \frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}$

iii. $\vec{RT} = \vec{OT} - \vec{OR}$

$\vec{RT} = 2/3\,\vec{RS}$

$= 2/3\left(\vec{OS} - \vec{OR}\right)$

$= 2/3\left(\left[\vec{OC} + \frac{1}{3}\vec{CA}\right] - \left[\vec{OB} + 2/3\,\vec{BA}\right]\right)$

$= 2/3\left(\underset{\sim}{c} + \frac{1}{3}(\underset{\sim}{a} - \underset{\sim}{c}) - \underset{\sim}{b} - 2/3\,(\underset{\sim}{a} - \underset{\sim}{b})\right)$

$= 2/3\left(2/3\,\underset{\sim}{c} - 1/3\,\underset{\sim}{a} - 1/3\,\underset{\sim}{b}\right)$

$= 4/9\,\underset{\sim}{c} - 2/9\,\underset{\sim}{a} - 2/9\,\underset{\sim}{b}$

iv. Collinear:

Prove $\vec{BT} = \lambda\vec{BM}$

$\vec{BT} = \lambda\left(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}\right)$ (from part ii)

Now $\vec{BT} = \vec{BR} + \vec{RT}$

$= -2/3\,\vec{AB} + \vec{RT}$

$= -\frac{2}{3}(\underset{\sim}{b} - \underset{\sim}{a}) + \left(\frac{4}{9}\underset{\sim}{c} - \frac{2}{9}\underset{\sim}{a} - \frac{2}{9}\underset{\sim}{b}\right)$ (from part iii)

$= -\frac{8}{9}\underset{\sim}{b} + \frac{4}{9}\underset{\sim}{c} + \frac{4}{9}\underset{\sim}{a}$

$= \frac{4}{9}(\underset{\sim}{a} - 2\underset{\sim}{b} + \underset{\sim}{c})$

$= \frac{8}{9}\left(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}\right)$

$= \frac{8}{9}\overrightarrow{BM}$

$\therefore$ T lies on $\overrightarrow{BM}$.